Quantum Martingale Theory and Entropy Production

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We employ martingale theory to describe fluctuations of entropy production for open quantum systems in nonequilbrium steady states. Using the formalism of quantum jump trajectories, we identify a decomposition of entropy production into an exponential martingale and a purely quantum term, both obeying integral fluctuation theorems. An important consequence of this approach is the derivation of a set of genuine universal results for stopping-time and infimum statistics of stochastic entropy production. Finally, we complement the general formalism with numerical simulations of a qubit system.

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The development of stochastic thermodynamics in recent decades allowed the description of work, heat, and entropy production associated with single trajectories in nonequilibrium processes [1,2]. This framework has successfully provided several genuine insights on the second law, such as the discovery of universal relations constraining the statistics of fluctuating thermodynamic quantities, usually known as fluctuation theorems [3–5]. The fundamental interest in refining our understanding of irreversibility and their microscopic imprints has been brought to its ultimate consequences by extending stochastic thermodynamics to the quantum realm [6], where fluctuation theorems have been derived [7–13] and experimentally tested in recent years [14,15].

When information about entropy production in single trajectories of a process is available, a natural question to ask is to what extent this information can be useful, for instance, whether or not it is possible to implement strategies leading to a reduction in entropy which might be eventually used as a fuel, like in the celebrated Maxwell's demon [16,17]. In the same context, one may ask whether the second law of thermodynamics will manifest as fundamental constraints limiting such strategies. A powerful method to handle these general questions is to employ a set of particularly interesting stochastic processes—namely, martingales [18]. Martingales are well known in mathematics [19] and quantitative finance as models of fair financial markets [20]. However, martingale theory has been only lightly exploited until now both in stochastic thermodynamics [21–27] and quantum physics [28,29].

Applying concepts of martingale theory in stochastic thermodynamics, it has been shown that the exponential of minus the entropy production $\Delta S_{\mathrm{tot}}(t)$ associated with classical trajectories $\gamma_{\{0,t\}}$—single paths in phase space—in generic nonequilibrium steady-state conditions is an *exponential martingale*, i.e., $\langle e^{-\Delta S_{\mathrm{tot}}(t)} | \gamma_{\{0,\tau\}} \rangle = e^{-\Delta S_{\mathrm{tot}}(\tau)}$, for any $t \geq \tau \geq 0$, where $\langle X(t) | \gamma_{\{0,\tau\}} \rangle$ denotes the conditional expectation of a functional $X(t)$ given $\gamma_{\{0,\tau\}}$ [19]. It has been shown that the martingality of $e^{-\Delta S_{\mathrm{tot}}(t)}$ implies a series of universal equalities and inequalities concerning the statistics of infima and stopping times (e.g., first passage, escape times, etc.) of entropy production [21,26].

In this Letter, we generalize the martingale theory of entropy production—so far developed only for classical systems—to the context of quantum thermodynamics, using the formalism of quantum jump trajectories [30,31]. Genuine quantum effects (such as coherence [32–34] and quantum correlations [35–37]) introduce new qualitative features that radically modify the framework. Indeed, dealing with martingales and stopping times requires conditioning on past events, which entails several difficulties in the quantum realm. Since evaluating stochastic entropy production along quantum trajectories $\gamma_{\{0,t\}}$ requires a two-point measurement protocol using direct measurements on the system [9,12,38–43], the development of a suitable conditioning not disturbing the dynamics becomes challenging. See Fig. 1 for an illustration.

In the following, we introduce a new auxiliary entropy production $\Delta S_{\mathrm{mar}}(t)$ and show that $e^{-\Delta S_{\mathrm{mar}}(t)}$ is a martingale along quantum trajectories [Eq. (5)], while in general $e^{-\Delta S_{\mathrm{tot}}(t)}$ is not. We use this finding to derive several universal relations valid for all nonequilibrium steady states, providing new insights on the role of uncertainty and coherence in the second law. Our key results are established in Eqs. (5)–(12). They comprise (i) a genuine quantum-classical decomposition of stochastic entropy production as the sum of two quantities, both fulfilling integral fluctuation theorems [Eqs. (6) and (7)], entailing a tight lower bound for the average entropy production [Eq. (8)]; (ii) fluctuation theorems and inequalities for

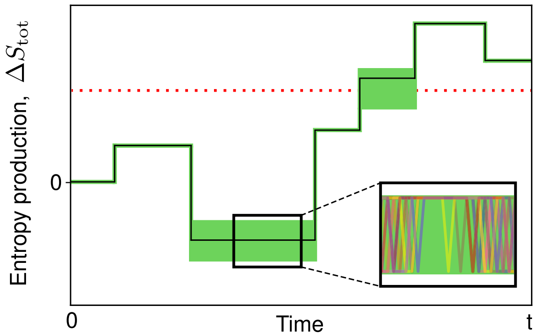

FIG. 1. Sample traces of stochastic entropy production in a nonequilibrium stationary process as a function of time. We show the entropy production $\Delta S_{\rm tot}$ associated with classical (black thin line) and quantum (thick green line) trajectories of duration t. When the system state becomes a superposition, $\Delta S_{\rm tot}$ is not uniquely defined in the quantum case (thick green segments) but only at the final time of the evolution t, when a direct measurement on the system is performed. (Inset) Different values that $\Delta S_{\rm tot}$ would take if system measurements were performed at intermediate times, for a given record of measurements in the environment. How can one then determine stopping times? For example, when does entropy production reach a threshold (red dotted line) for the first time in an open quantum system?

stopping-time statistics of entropy production [Eqs. (9) and (10)]; and (iii) inequalities for the extreme-value statistics of entropy production that generalize to the quantum realm the results derived in Refs. [21,22] [Eqs. (11) and (12)].

Entropy production in quantum trajectories.—One of the most successful approaches describing the stochastic thermodynamics of open quantum systems is the formalism of quantum jump trajectories [38–51]. This formalism describes the stochastic evolution of the pure state of the system $|\psi(t)\rangle$, conditioned on measurements obtained from the continuous monitoring of the environment [30,31].

Within this approach, the system evolution is described as a smooth evolution intersected by quantum jumps in the state of the system occurring at random times. These jumps correspond to the detection of different type of events in the environment (e.g., the emission or absorption of energy quanta from different thermal reservoirs) leading to a measurement record $\mathcal{R}_0^t = \{(k_1, t_1), \ldots, (k_J, t_J)\}$, where (k_j, t_j) indicates that a jump of type k_j occurred at time t_j, $j = 1, \ldots, J$, and $0 \leq t_1 \leq \cdots \leq t_J \leq t$. The evolution is described by the stochastic Schrödinger equation

$$d|\psi(t)\rangle = dt\left(-\frac{i}{\hbar}H + \sum_k \frac{\langle L_k^\dagger L_k\rangle_{\psi(t)} - L_k^\dagger L_k}{2}\right)|\psi(t)\rangle + \sum_k dN_k(t)\left(\frac{L_k}{\sqrt{\langle L_k^\dagger L_k\rangle_{\psi(t)}}} - \mathbb{1}\right)|\psi(t)\rangle, \quad (1)$$

where H is a Hermitian operator (usually the system Hamiltonian), L_k for $k = 1 \ldots K$ are the Lindblad (jump) operators, and here and in the following we denote as $\langle A\rangle_{\psi(t)} \equiv \langle\psi(t)|A|\psi(t)\rangle$ quantum-mechanical expectation values, and $\mathbb{1}$ as the identity matrix. The random variables $dN_k(t)$ are Poisson increments associated with the number of jumps $N_k(t)$ of type k detected up to time t in the process, leading to the record $\mathcal{R}$. The $dN_k(t)$ take most of the time the value 0, becoming 1 only at times t_j when a jump of type k_j is detected in the environment. When averaging over measurement outcomes, the evolution reduces to a Markovian process ruled by a Lindblad master equation [52,53].

From now on, we consider nonequilibrium steady states. Here, the initial state of the trajectories is sampled from the spectral decomposition of the steady state of the master equation, $\pi = \sum_n \pi_n|\pi_n\rangle\langle\pi_n|$. We also require a two-measurement protocol, where projective measurements in the π eigenbasis are executed at the beginning $(t = 0)$ and at the end $(t = \tau)$ of any single trajectory. For this setup, a probability $P(\gamma_{\{0,\tau\}})$ can be associated with any trajectory $\gamma_{\{0,\tau\}} = \{n(0); \mathcal{R}_0^\tau; n(\tau)\}$, where $n(0)$ and $n(\tau)$ are the outcomes of the first and final measurements on the system. The entropy production (for $k_B = 1$) associated with the trajectory $\gamma_{\{0,\tau\}}$ is defined as the functional

$$\Delta S_{\rm tot}(\tau) \equiv \ln\frac{P(\gamma_{\{0,\tau\}})}{\tilde{P}(\tilde{\gamma}_{\{0,\tau\}})} = \ln\left(\frac{\pi_{n(0)}}{\pi_{n(\tau)}}\right) + \sum_{j=1}^{J}\Delta S_{\rm env}^{k_j}, \quad (2)$$

where $\tilde{P}(\tilde{\gamma}_{\{0,\tau\}})$ is the probability of the time-reversed trajectory $\tilde{\gamma}_{\{0,\tau\}} = \{n(\tau); \tilde{R}_\tau^0; n(0)\}$, with $\tilde{\mathcal{R}}_\tau^0 = \{(k_J, t_J), \ldots, (k_1, t_1)\}$ being the time-reversed sequence of jumps, occurring in the time-reversed (or backward) process [12,38,39]. In Eq. (2), the first term in the right hand side (rhs) is the system entropy change along the trajectory [1], and $\Delta S_{\rm env}^{k_j}$ is the environmental entropy change due to the jump k_j, which in most cases of physical interest obeys the local detailed balance condition for pairs of operators $L_k = L_{k'}^\dagger e^{\Delta S_{\rm env}^k/2}$ [39]. The averages of both terms yield the von Neumann entropy changes of system and environment, respectively [12]. The stochastic entropy production given by Eq. (2) obeys the integral fluctuation theorem $\langle e^{-\Delta S_{\rm tot}(\tau)}\rangle = 1$, which leads to the second-law inequality $\langle\Delta S_{\rm tot}(\tau)\rangle \geq 0$. The classical limit is recovered when the stochastic wave function $|\psi(t)\rangle$ obtained from Eq. (1) is an eigenstate of π at any time of the dynamical evolution [54].

Quantum martingale theory.—The main ingredient for the development of our quantum martingale theory is the definition of conditional averages over trajectories with common history up to a certain time $\tau \leq t$. To this end, one needs to define both $\gamma_{\{0,\tau\}}$ and $\gamma_{\{0,t\}}$. Here, unlike for classical trajectories, $\gamma_{\{0,\tau\}} \not\subseteq \gamma_{\{0,t\}}$ because $\gamma_{\{0,\tau\}}$ includes a measurement at time τ while $\gamma_{\{0,t\}}$ does not, and therefore entropy production for trajectory $\gamma_{\{0,t\}}$ is not well defined at time τ (see Fig. 1). Thus, we define the conditional average of a generic stochastic process $X(t)$ defined along a trajectory $\gamma_{\{0,t\}}$ as $\langle X(t)|\gamma_{[0,\tau]}\rangle = \sum_{n(t),\mathcal{R}_\tau^t} X(t)P(\gamma_{\{0,t\}}|\gamma_{[0,\tau]})$, where we condition with respect to the *ensemble* of trajectories $\gamma_{[0,\tau]} \equiv \bigcup_{s=0}^{\tau}\gamma_{\{0,s\}}$ that includes all of the outcomes of trajectories eventually stopped (i.e., measured) at all intermediate times in the interval $[0,\tau]$. Note that in the classical limit the ensemble $\gamma_{[0,\tau]} = \gamma_{\{0,\tau\}}$ contains just one trajectory. Furthermore, since $P(\gamma_{\{0,t\}}|\gamma_{[0,\tau]}) = P(\gamma_{\{0,t\}}|\gamma_{\{0,\tau\}})$ (see the Supplemental Material [55]), then $\langle X(t)|\gamma_{[0,\tau]}\rangle = \langle X(t)|\gamma_{\{0,\tau\}}\rangle$.

We now discuss quantum martingales in relation to stochastic entropy production given by Eq. (2). Notably, unlike for classical systems [21,22], the process $e^{-\Delta S_{\rm tot}(t)}$ is not a martingale in this context because $\langle e^{-\Delta S_{\rm tot}(t)}|\gamma_{[0,\tau]}\rangle = e^{-\Delta S_{\rm tot}(\tau)+\Delta S_{\rm unc}(\tau)}$ for all $t \geq \tau \geq 0$, with

$$\Delta S_{\rm unc}(t) \equiv -\ln\left(\frac{\pi_{n(t)}}{\langle\pi\rangle_{\psi(t)}}\right); \tag{3}$$

see Ref. [55] for a detailed proof. The origin of $\Delta S_{\rm unc}$ can be traced back to the quantum uncertainty in the evolution. It measures how informative (surprising) is the occurrence of outcome $n(t)$ with respect to the average result when measuring the stochastic wave function $|\psi(t)\rangle$ at that time, that is, $\langle\pi\rangle_{\psi(t)} = \sum_i \pi_i |\langle\pi_i|\psi(t)\rangle|^2$. The quantity $\langle\pi\rangle_{\psi(t)}$ is the squared Uhlman's fidelity between states π and $|\psi(t)\rangle$, which quantifies the distinguishably of these two states. In other words, $\Delta S_{\rm unc}(t)$ measures how much information we gain knowing the outcome $n(t)$ of the measurement at time t with respect to knowing only $|\psi(t)\rangle$. Importantly, the "uncertainty" entropy production $\Delta S_{\rm unc}(t)$ satisfies the property $\langle e^{-\Delta S_{\rm unc}(t)}|\gamma_{[0,\tau]}\rangle = 1$ for any $\tau \leq t$ [55], and it is bounded at all times by $\pi_{\rm min}/\pi_{\rm max} \leq e^{-\Delta S_{\rm unc}(t)} \leq \pi_{\rm max}/\pi_{\rm min}$, where $\pi_{\rm min} = \min_i \pi_i$ and $\pi_{\rm max} = \max_i \pi_i$ are the minimum and maximum eigenvalues of π. In the classical limit $\langle\pi\rangle_{\psi(t)} = \pi_{n(t)}$ in Eq. (3), leading to $\Delta S_{\rm unc}(t) = 0$ at any time t, and we recover the classical result.

We now define the auxiliary ("martingale") entropy production as $\Delta S_{\rm mar}(t) \equiv \Delta S_{\rm tot}(t) - \Delta S_{\rm unc}(t)$, which using Eqs. (2) and (3) gives

$$\Delta S_{\rm mar}(t) = \ln\left(\frac{\pi_{n(0)}}{\langle\pi\rangle_{\psi(t)}}\right) + \sum_{j=1}^{J} \Delta S_{\rm env}^{k_j}, \tag{4}$$

to be compared with Eq. (2). Note that $\Delta S_{\rm mar}(t)$ results from replacing $\pi_{n(t)}$ with $\langle\pi\rangle_{\psi(t)}$ in the boundary term in Eq. (2), therefore avoiding the need for information from measurements at time t. We prove that $\Delta S_{\rm mar}(t)$ is an exponential martingale:

$$\langle e^{-\Delta S_{\rm mar}(t)}|\gamma_{[0,\tau]}\rangle = e^{-\Delta S_{\rm mar}(\tau)}, \tag{5}$$

which holds for any $t \geq \tau \geq 0$ [55]. Recall that the average in Eq. (5) is conditioned over the ensemble of trajectories $\gamma_{[0,\tau]}$ containing all possible measurement outcomes in both system and environment at times smaller than τ. However, since $\langle X(t)|\gamma_{[0,\tau]}\rangle = \langle X(t)|\gamma_{\{0,\tau\}}\rangle$ for any functional $X(t)$ of $\gamma_{\{0,t\}}$, one also has a martingale condition with respect to single quantum trajectories, $\langle e^{-\Delta S_{\rm mar}(t)}|\gamma_{\{0,\tau\}}\rangle = e^{-\Delta S_{\rm mar}(\tau)}$.

Martingale fluctuation theorems and second law.—The first consequence of Eq. (5), together with the properties of $\Delta S_{\rm unc}(t)$ in Eq. (3), is the following decomposition of stochastic entropy production,

$$\Delta S_{\rm tot}(t) = \Delta S_{\rm unc}(t) + \Delta S_{\rm mar}(t), \tag{6}$$

with both summands in Eq. (6) fulfilling an integral fluctuation theorem:

$$\langle e^{-\Delta S_{\rm unc}(t)}\rangle = 1, \qquad \langle e^{-\Delta S_{\rm mar}(t)}\rangle = 1. \tag{7}$$

Notably, this decomposition has the same structure as the Oono-Paniconi [57] (and adiabatic-nonadiabatic [58]) decomposition of entropy production in nonequilibrium systems [59]. From Jensen's inequality $\langle e^x\rangle \geq e^{\langle x\rangle}$ and Eq. (7), we have both $\langle\Delta S_{\rm unc}(t)\rangle \geq 0$ and $\langle\Delta S_{\rm mar}(t)\rangle \geq 0$. Moreover, using Eq. (6) and the second law, we obtain the following bound for the average entropy production:

$$\langle\Delta S_{\rm tot}(t)\rangle \geq \langle\Delta S_{\rm mar}(t)\rangle. \tag{8}$$

We remark that the bound (8) allows us to estimate the average entropy production without the need for any measurement on the system at intermediate times. This inequality provides a tight bound because $\Delta S_{\rm unc}(t)$ is bounded and thus not extensive in time [60]. In the classical limit and when approaching equilibrium conditions $\Delta S_{\rm unc}(t) = 0$, and Eq. (8) becomes an equality.

Stopping-time statistics.—An important result in martingale theory is Doob's optional stopping theorem [61], which concerns stopping-time statistics of martingales. A paradigmatic example of a stopping time $\mathcal{T}$ is the first time at which a stochastic process $X(t)$ reaches a subset $\mathcal{X}$ of the state space. Importantly, a stopping time $\mathcal{T}$ is a random variable whose value can be determined solely by looking at the past history of the process $X_{[0,\mathcal{T}]}$. Applying Doob's optional stopping theorem to the martingale $e^{-\Delta S_{\rm mar}(t)}$, one finds

$$\langle e^{-\Delta S_{\rm mar}(\mathcal{T})}\rangle = 1; \tag{9}$$

i.e., its average over the stopping times $\mathcal{T}$ equals its average value at the initial time $t = 0$ [55]. Again, using Jensen's inequality and Eq. (9), we obtain a second-law-like inequality at stopping times $\langle\Delta S_{\rm mar}(\mathcal{T})\rangle \geq 0$ which implies

$$\langle\Delta S_{\rm tot}(\mathcal{T})\rangle \geq \langle\Delta S_{\rm unc}(\mathcal{T})\rangle. \tag{10}$$

Note that here $\mathcal{T}$ are stopping times defined in terms of $\gamma_{[0,\mathcal{T}]}$ which are well defined, e.g., when the stopping condition uses the auxiliary process $\Delta S_{\rm mar}(t)$. In the classical limit $\Delta S_{\rm unc}(t) = 0$, Eq. (9) reduces to the integral fluctuation theorem for entropy production at stopping times $\langle e^{-\Delta S_{\rm tot}(\mathcal{T})}\rangle = 1$, and Eq. (10) to the second law $\langle \Delta S_{\rm tot}(\mathcal{T})\rangle \geq 0$. We remark that $\langle \Delta S_{\rm unc}(\mathcal{T})\rangle$ can in principle be either positive or negative. Therefore, Eq. (10) does not exclude the possibility that the average entropy production at stopping times may be *negative* for particular choices of stopping times.

Extreme-value statistics.—Very recently, universal statistics of infima of stochastic entropy production have been unveiled [21] using Doob's maximal inequality [61], which bounds the probability of the supremum of a positive martingale process $M(t)$ as $\Pr[\sup_{\tau\in[0,t]} M(\tau) \geq \lambda] \leq \langle M(t)\rangle/\lambda$ for $\lambda \geq 0$. Applying Doob's maximal inequality to the positive martingale $e^{-\Delta S_{\rm mar}(t)}$, we derive the following inequality for the probability that the finite-time infimum $\inf_{\tau\in[0,t]} \Delta S_{\rm mar}(\tau) \geq 0$ lies below a certain value,

$$\Pr\left[\inf_{\tau\in[0,t]} \Delta S_{\rm mar}(\tau) \leq -\xi\right] \leq e^{-\xi}, \qquad (11)$$

where $\xi \geq 0$ and $t \geq 0$ [55]. Therefore, the probability of observing extreme reductions of the martingale entropy production along single trajectories of any duration t is exponentially suppressed. As a consequence, the average infimum of the martingale entropy production obeys $\langle \inf_{\tau\in[0,t]} \Delta S_{\rm mar}(\tau)\rangle \geq -1$, which generalizes the infimum law for classical nonequilibrium stationary states [21]. From Eq. (11) and the condition $\Delta S_{\rm unc}(t) \leq \ln(\pi_{\rm max}/\pi_{\rm min})$ for all $t \geq 0$, we derive

$$\left\langle \inf_{\tau\in[0,t]} \Delta S_{\rm tot}(\tau) \right\rangle \geq -1 - \ln\left(\frac{\pi_{\rm max}}{\pi_{\rm min}}\right). \qquad (12)$$

Since $\ln(\pi_{\rm max}/\pi_{\rm min}) \geq 0$, Eq. (12) provides a lower bound for the average entropy-production infimum that is below the infimum law for classical systems $\langle \inf_{\tau\in[0,t]} \Delta S_{\rm tot}(\tau)\rangle \geq -1$.

Quantum martingale theory at work.—We conclude by illustrating our theory with a simple example, amenable to a direct experimental realization. Our model consists of a single spin-1/2 particle (a qubit) with Hamiltonian $H = \hbar\omega\sigma_z/2$ which is subjected to two different and orthogonal sources of noise. The dynamical evolution is described by the stochastic Schrödinger equation (1), where a monitoring process detects the jumps induced in the system by both noise sources. This can be visualized in the Bloch sphere where the stochastic wave function sits at all times during its evolution [see Fig. 2(a)]. The first source corresponds to thermal noise, which produces jumps in the z direction of the particle as described by Lindblad operators $L_\downarrow = \sqrt{\gamma_\downarrow}\sigma^-$ and $L_\uparrow = \sqrt{\gamma_\uparrow}\sigma^+$, with rates fulfilling local detailed balance $\gamma_\downarrow = \gamma_\uparrow e^{-\beta\hbar\omega}$, with β being the inverse temperature. The environmental entropy changes associated with these jumps are, respectively, $\Delta S_{\rm env}^{\downarrow\uparrow} = \pm\beta\omega$. Furthermore,

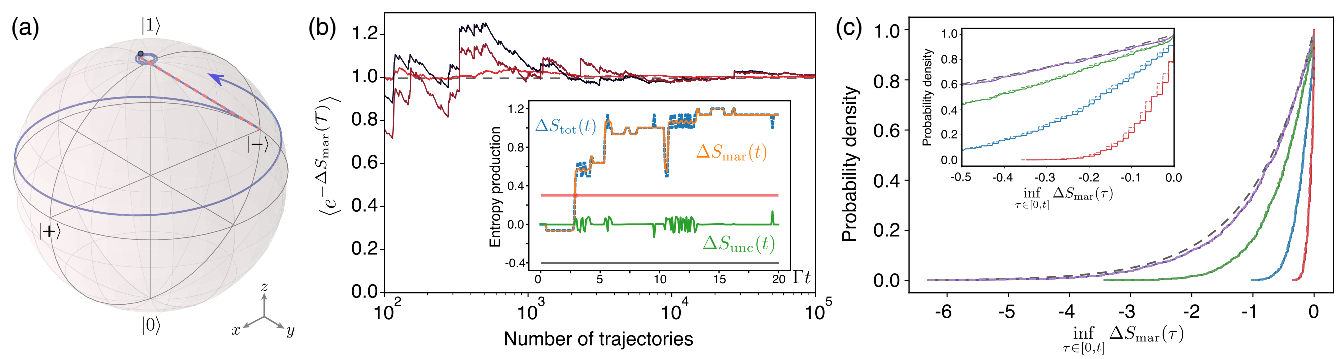

FIG. 2. (a) Sample trajectory of $|\psi(t)\rangle$ represented in the qubit's Bloch sphere. The smooth evolution starting from an eigenstate (blue circle) is interrupted by a jump L_- (dotted line) which abruptly collapses the qubit state to $|-\rangle = [|0\rangle - |1\rangle]/\sqrt{2}$ along the x axis, after which the evolution continues up to the point marked by the arrow. (b) Convergence of the stopping-time fluctuation theorem for $\Delta S_{\rm mar}$ as a function of the number of trajectories for three different stopping rules: stopping at a fixed time $t_f \equiv 20\Gamma^{-1}$ (black line); $\min(\mathcal{T}_1, t_f)$ with $\mathcal{T}_1$ the first-passage time of $\Delta S_{\rm mar}$ to reach a positive threshold located at 0.3 (brown line); and $\min(\mathcal{T}_2, t_f)$, with $\mathcal{T}_2$ being the unconditional first-passage time of $\Delta S_{\rm mar}$ to reach either the thresholds 0.3 or −0.4 (red line). The dashed gray line in 1 is a guide for the eye. (Inset) Sample traces of stochastic entropy production (blue dashed line) and its decomposition as the sum of $\Delta S_{\rm mar}$ (orange solid line) plus $\Delta S_{\rm unc}$ (green solid line). The horizontal thick lines are the two absorbing boundaries used to compute stopping times. The parameters of the simulation are $\hbar\omega = 1$, $\beta = 0.2$, $\eta = 0.5$, $\Gamma \equiv \gamma_- - \gamma_+ = \gamma_\downarrow - \gamma_\uparrow = 0.01$. (c) Empirical cumulative distributions of the finite-time minimum of $\Delta S_{\rm mar}$ (solid lines) and $\Delta S_{\rm tot}$ (dashed lines) for different values of the observation time: $\Gamma t = 10$ (red), $\Gamma t = 10^2$ (blue), $\Gamma t = 10^3$ (green), and $\Gamma t = 10^4$ (purple). The gray dashed line is the exponential in the right-hand side of Eq. (11). (Inset) Enlarged view of the distribution for small values of the infima. The parameters for the simulation are $\hbar\omega = 1$, $\beta = \eta = 0.04$, $\Gamma = 0.01$.

a second source of noise generates jumps in the x direction induced by the Lindblad operators $L_- = \sqrt{\gamma_-}(\sigma_z - i\sigma_y)/2$ and $L_+ = \sqrt{\gamma_+}(\sigma_z + i\sigma_y)/2$, with $\gamma_- = \gamma_+ e^{-\eta}$. Notice that here η is a bias parameter that plays the role of an inverse temperature in the x direction. Analogously, the entropy changes associated with such jumps are $\Delta S_{\rm env}^{\mp} = \pm\eta$. The dynamical evolution in this setup is genuinely quantum. Since the two sets of jumps occur in orthogonal directions, the generation of superpositions of π eigenstates along the quantum trajectories is guaranteed. The classical limit is recovered for high temperatures ($\beta \to 0$) and large bias ($\eta \to 0$), where the steady state becomes the maximally mixed state $\pi \to \mathbb{1}/2$, corresponding to the equilibrium distribution.

We performed numerical simulations of the qubit system using quantum-trajectory Monte Carlo methods [62]. We find perfect agreement between our simulations and the stopping-time fluctuation theorem in Eq. (9), which is verified by three different stopping times $\mathcal{T}$ [Fig. 2(b)]. Interestingly, the convergence to the theoretical value is much faster for first-passage times over thresholds (see inset) than using a fixed (final) time as in the standard integral fluctuation theorem. This makes the stopping-time fluctuation theorem (9) more amenable for experimental tests. We then evaluate statistics of the finite-time minimum of $\Delta S_{\rm mar}$ and $\Delta S_{\rm tot}$. The tails of the distribution of the finite-time minima of $\Delta S_{\rm mar}$ are exponentially suppressed in agreement with Eq. (11) [Fig. 2(c)]. Moreover, the average minima of both $\Delta S_{\rm mar}$ and $\Delta S_{\rm tot}$ lie above -1 for this choice of parameters.

Discussion.—Our work shows that martingale theory can be generalized to quantum thermodynamics providing insights about entropy production beyond fluctuation theorems. Here, we provided new nonequilibrium universal relations along quantum trajectories [Eqs. (5)–(12)]. Our results may be of particular importance in setups allowing environmental monitoring and feedback control [50,51,63–68], and for quantum thermal devices working in nonequilibrium steady-state conditions [69–76]. It would also be interesting to explore connections with path-integral approaches [77], one-shot quantum thermodynamics [78–81], and quantum information [82–86]. Finally, we remark that some of our results could be applied to classical systems where knowledge of a system's state is incomplete, e.g., under coarse graining of hidden internal microstates [87,88].

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[17] *Maxwell's Demon 2: Entropy, Classical and Quantum Information, Computing*, edited by H. Leff and A. F. Rex (CRC Press, Boca Raton, FL, 2002).

[18] A martingale process is a stochastic process defined on a probability space whose expected value at any time t equals its value at some previous time $\tau < t$ when conditioned on observations up to that time τ. More formally, $M(t)$ is a martingale if it is bounded $\langle M(t)\rangle < \infty$ for all t and verifies $\langle M(t)|M_{\{0,\tau\}}\rangle = M(\tau)$, where the latter average is conditioned on all of the previous values $M_{\{0,\tau\}}$ of the process up to time τ [19].

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